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# Equations Governing the Propagation of Second-Order Correlations in Non-Stationary Electromagnetic Fields (*).

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**Summary.** — The main results derived by E. WOLF and P. ROMAN, relating to the propagation of second-order correlations in electromagnetic fields are generalized to the case of a non-stationary field containing currents and charges. The basic differential equations relating the correlations are derived. They fall into two groups, one of which contains only differential equations of the first order, but involves certain parameters that seem difficult to be determined experimentally. When these quantities are eliminated a second set of equations is obtained. Equations of this set are of a higher order but they contain only the electric and the magnetic correlation tensors and a tensor characterizing the correlation in the electric currents.

## 1. – Introduction.

In several papers published in recent years (1-4) a unified formulation of the theory of partial coherence and partial polarization was obtained, for stationary electromagnetic fields in vacuo. The chief mathematical tools in this theory are certain correlation tensors which describe the cross-correlation

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(1) E. WOLF: *Nuovo Cimento*, **12**, 884 (1954).

(2) E. WOLF: Contribution in *Proc. Symposium on Astronom. Optics* (Amsterdam, 1956), p. 177.

(3) P. ROMAN and E. WOLF: *Nuovo Cimento*, **17**, 462 (1960). We shall refer to reference (3) as paper I. Formulae from this paper will be denoted as (I.3.11a) etc.

(4) P. ROMAN and E. WOLF: *Nuovo Cimento*, **17**, 477 (1960).

functions of the electromagnetic field at any two points in the field. These correlations tensors were found to be rigorously connected by a set of partial linear differential equations. From these equations a number of conservation laws were derived. Some applications of the theory were described in reference (5) and (6).

More recently ROMAN (7) extended the theory to take into account the presence of random charges and currents and he suggested possible application of the theory to the study of propagation of electromagnetic fields in plasmas. Another possible application of the theory may well be in studies of the properties of the electromagnetic field generated by an optical maser. For the purpose of these and other applications (*e.g.* the analysis of transient phenomena), it is, however, desirable to extend the theory to electromagnetic fields that are not necessarily stationary. It is the purpose of the present paper to derive the basic differential equations relating to fields of this type.

In Section **2** the definition of the basic correlation functions is extended to non-stationary electromagnetic fields containing charges and currents. In Section **3** a set of partial differential equations governing the propagation of these quantities in vacuo is derived. These equations are again linear partial differential equations of the first order but they include certain quantities that appear to be difficult to determine experimentally. These quantities are eliminated in Section **4** and one is led to linear partial differential equations of higher orders which, however, involve only the basic correlation tensors of the theory.

## 2. – The defining equations.

Let $E_j(\boldsymbol{x}, t)$ and $H_j(\boldsymbol{x}, t)$ ($j = 1, 2, 3$) be the analytic signals (8) associated with the Cartesian components of the electric and magnetic field vectors at a typical field point $P(\boldsymbol{x})$ at time $t$. We define the ***field correlation tensors*** for the non-stationary case by the relations

$$
(2.1)\qquad \left\{
\begin{aligned}
\mathscr{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle E_j(\boldsymbol{x}_1, t+\tau_1)\, E_k^*(\boldsymbol{x}_2, t+\tau_2)\rangle ,\\
\mathscr{H}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle H_j(\boldsymbol{x}_1, t+\tau_1)\, H_k^*(\boldsymbol{x}_2, t+\tau_2)\rangle ,\\
\mathscr{G}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle E_j(\boldsymbol{x}_1, t+\tau_1)\, H_k^*(\boldsymbol{x}_2, t+\tau_2)\rangle ,\\
\widetilde{\mathscr{G}}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle H_j(\boldsymbol{x}_1, t+\tau_1)\, E_k^*(\boldsymbol{x}_2, t+\tau_2)\rangle ,
\end{aligned}
\right.
\qquad (j, k = 1, 2, 3) .
$$

(5) E. WOLF: *Nuovo Cimento*, **13**, 1165 (1959).
(6) G. B. PARRENT and P. ROMAN: *Nuovo Cimento*, **15**, 370 (1960).
(7) P. ROMAN: *Nuovo Cimento*, **20**, 759 (1961).
(8) M. BORN and E. WOLF: *Principles of Optics* (London and New York), § 10.2.

Here the sharp brackets denote time average, defined as follows:

$$(2.2)\qquad \langle F(t+\tau_1)\,G(t+\tau_2)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-\infty}^{+\infty} F(t+\tau_1;\,T)\,G(t+\tau_2;\,T)\,dt\,,$$

with

$$F(t,T) = F(t) \qquad |t| < T.$$
$$= 0 \qquad |t| > T.$$

In this special case when the field is stationary, the quantities defined by (2.1) each become functions of the difference $\tau=\tau_1-\tau_2$ only and the four correlation tensors reduce to those introduced by WOLF [1] and discussed fully by him and ROMAN [3-5,7].

As is easily seen from the definitions these tensors have the following symmetry relations

$$(2.3)\qquad \begin{cases} \mathscr{E}_{jk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau_1,\tau_2) = \mathscr{E}^*_{kj}(\boldsymbol{x}_2,\boldsymbol{x}_1,\tau_2,\tau_1)\,,\\ \mathscr{H}_{jk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau_1,\tau_2) = \mathscr{H}^*_{jk}(\boldsymbol{x}_2,\boldsymbol{x}_1,\tau_2,\tau_1)\,,\\ \mathscr{G}_{jk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau_1,\tau_2) = \widetilde{\mathscr{G}}^*_{kj}(\boldsymbol{x}_2,\boldsymbol{x}_1,\tau_2,\tau_1)\,,\\ \widetilde{\mathscr{G}}_{jk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau_1,\tau_2) = \mathscr{G}^*_{kj}(\boldsymbol{x}_2,\boldsymbol{x}_1,\tau_2,\tau_1)\,. \end{cases}$$

Further we define a *change correlation scalar* by the relation

$$(2.4)\qquad \mathscr{P}(x_1,x_2,\tau_1,\tau_2) = \langle\varrho(\boldsymbol{x}_1,t+\tau)\,\varrho^*(\boldsymbol{x}_2,t+\tau_2)\rangle\,,$$

where $\varrho(\boldsymbol{x},t)$ is the analytic signal associated with the real change density.

Finally we introduce the *current correlation tensor* $\mathscr{I}$ by the formula

$$(2.5)\qquad \mathscr{I}_{jk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau_1,\tau_2) = \langle i_j(\boldsymbol{x}_1,t+\tau_1)\,i^*_k(\boldsymbol{x}_2,t+\tau_2)\rangle\,,$$

where $i_j$ $(j=1,2,3)$ is the analytic signal associated with a typical Cartesian components of the electric current vector.

Obviously $\mathscr{P}$ and $\mathscr{I}$ have the following symmetry relations:

$$(2.6)\qquad \begin{cases} \mathscr{P}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau_1,\tau_2) = \mathscr{P}^*(\boldsymbol{x}_2,\boldsymbol{x}_1,\tau_2,\tau_1)\,,\\ \mathscr{I}_{jk}(\boldsymbol{x}_1,\boldsymbol{x}_2,\tau_1,\tau_2) = \mathscr{I}^*_{kj}(\boldsymbol{x}_2,\boldsymbol{x}_1,\tau_2,\tau_1)\,. \end{cases}$$

In addition we introduce, for notational convenience, the auxiliary correlations

$$
(2.7)\qquad \left\{
\begin{aligned}
\alpha_j(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle E_j(\boldsymbol{x}_1, t+\tau_1)\, \varrho^*(\boldsymbol{x}_2, t+\tau_2)\rangle\,,\\
\beta_j(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle H_j(\boldsymbol{x}_1, t+\tau_1)\, \varrho^*(\boldsymbol{x}_2, t+\tau_2)\rangle\,,\\
\gamma_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle E_j(\boldsymbol{x}_1, t+\tau_1)\, i^*_k(\boldsymbol{x}_2, t+\tau_2)\rangle\,,\\
\eta_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) &= \langle H_j(\boldsymbol{x}_1, t+\tau_1)\, i^*_k(\boldsymbol{x}_2, t+\tau_2)\rangle\,.
\end{aligned}
\right.
$$

The quantities $\mathscr{P}$, $\mathscr{J}_{jk}$, $\alpha_j$, $\beta_j$, $\gamma_{jk}$ and $\eta_{jk}$ are generalizations to non-stationary fields of similar quantities introduced recently by ROMAN (7).

## 3. – First order differential equations for the correlations.

The Maxwell equations in the presence of sources may be written in the form

$$
(3.1)\qquad \varepsilon_{jkl}\partial^1_k E_l(\boldsymbol{x}_1, t_1) + \frac{1}{c}\frac{\partial}{\partial t_1} H_j(\boldsymbol{x}_1, t_1) = 0\,,
$$

$$
(3.2)\qquad \varepsilon_{jkl}\partial^1_k H_l(\boldsymbol{x}_1, t_1) - \frac{1}{c}\frac{\partial}{\partial t_1} E_j(\boldsymbol{x}_1, t_1) = \frac{4\pi}{c}\, i_j(\boldsymbol{x}_1, t_1)\,,
$$

$$
(3.3)\qquad \partial^1_j E_j(\boldsymbol{x}_1, t_1) = 4\pi\varrho(\boldsymbol{x}_1, t_1)\,,
$$

$$
(3.4)\qquad \partial^1_j H_j(\boldsymbol{x}_1, t_1) = 0\,.
$$

where $\varepsilon_{jkl}$ is the completely antisymmetric unit tensor of Levi-Civita, *i.e.* $\varepsilon_{jkl}$ is $+1$ or $-1$ according as the subscripts $(j, k, l)$ are an even or an odd permutation of $(1, 2, 3)$ and $\varepsilon_{jkl} = 0$ when two suffices are equal. We shall use superscript 1 or 2 according whether the operator acts on the co-ordinates of $\boldsymbol{x}_1$, or $\boldsymbol{x}_2$, *i.e.*

$$
\partial^1_k = \frac{\partial}{\partial x^k_1}\,,\qquad \partial^2_k = \frac{\partial}{\partial x^k_2}\qquad (k = 1, 2, 3).
$$

If we multiply (3.3) by $\varrho^*(\boldsymbol{x}_2, t+\tau_2)$ and average over $t$, we obtain, if we also set $t_1 = t+\tau_1$

$$
\partial^1_j\langle E_j(\boldsymbol{x}_1, t+\tau_1)\, \varrho^*(\boldsymbol{x}_2, t+\tau_2)\rangle = 4\pi\langle\varrho(\boldsymbol{x}_1, t+\tau_1)\, \varrho^*(\boldsymbol{x}_2, t+\tau_2)\rangle\,.
$$

If we use the notation introduced in the preceding section, this relation becomes

$$
(3.5)\qquad \partial^1_j \alpha_j(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 4\pi\,\mathscr{P}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2)\,.
$$

Next we multiply (3.3) by $E_k^*(\boldsymbol{x}_2, t+\tau_2)$ and average over $t$. This gives the relation

$$\partial_j^1 \mathscr{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 4\pi \alpha_k^*(\boldsymbol{x}_2, \boldsymbol{x}_1, \tau_1, \tau_2)\,. \tag{3.6a}$$

If we start from the complex conjugate of (3.3) with $\boldsymbol{x}_1 \to \boldsymbol{x}_2$, $t_1 \to t_2$, $\partial^1 \to \partial^2$, multiply the equation by $E_j(\boldsymbol{x}_1, t_1)$, set $t_1 = t+\tau_1$, $t_2 = t+\tau_2$ and average over $t$, we obtain

$$\partial_k^2 \langle E_j(\boldsymbol{x}_1, t+\tau_1)\, E_k^*(\boldsymbol{x}_2, t+\tau_2)\rangle = 4\pi \langle E_j(\boldsymbol{x}_1, t+\tau_1)\, \varrho^*(\boldsymbol{x}_2, t+\tau_2)\rangle\,,$$

*i.e.*

$$\partial_k^2 \mathscr{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 4\pi \alpha_j(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2)\,. \tag{3.6b}$$

The two eqs. (3.6*a*) and (3.6*b*) are generalizations to a non-stationary field containing currents and charges of eqs. (I. 3.13*a*) and (I. 3.13*b*) of ROMAN and WOLF. These equations give the connection between $\mathscr{E}_{jk}$ and $\alpha_j$. It appears that $\mathscr{E}_{jk}$ is a measurable quantity. On the other hand $\alpha_j$ can hardly be measured, although it has a definite physical meaning, *i.e.* it represents the correlation between the charge density and the electric field.

Further we obtain from (3.3) in a similar way, the following two equations:

$$\partial_j^1 \mathscr{G}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 4\pi \beta_k^*(\boldsymbol{x}_2, \boldsymbol{x}_1, \tau_2, \tau_1)\,,$$

$$\partial_j^2 \tilde{\mathscr{G}}_{kj}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 4\pi \beta_k(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2)\,.$$

These equations cannot be considered as basic equations, since the quantities $\beta_k$ and $\beta_k^*$ are again not measurable. However, if the field is free, these equations turn out to be more meaningful. In that case $\beta_k = \beta_k^* = 0$ and the equations then represent divergence conditions on $\mathscr{G}$ and $\tilde{\mathscr{G}}$ and are generalizations of the simple divergence relations (I. 3.14*a*) and (I. 3.14*b*) for the stationary case.

From (3.4) we can readily obtain other divergence relations:

$$\partial_j^1 \tilde{\mathscr{G}}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 0\,, \tag{3.7a}$$

$$\partial_k^2 \mathscr{G}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 0\,, \tag{3.7b}$$

$$\partial_j^1 \mathscr{H}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 0\,, \tag{3.8a}$$

$$\partial_k^2 \mathscr{H}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 0\,. \tag{3.8b}$$

These four equations have respectively the same form as eqs. (I. 3.15*a*), (I. 3.15*b*),

(I. 3.6*a*) and (I. 3.16*b*). This is so because the starting eq. (3.4) from which they are derived is the same as in the source free case.

Next we shall derive the correlation equation which relate the correlation tensors $\mathscr{E}_{jk}$, $\mathscr{H}_{jk}$, $\eta_{jk}$, $\gamma_{jk}$ and $\mathscr{J}_{jk}$. These equations can be obtained from (3.2). Multiplying (3.2) by $i_n^*(\boldsymbol{x}_2, t+\tau_2)$ and using the same procedure of averaging as before we obtain the formulae

$$\varepsilon_{jkl}\partial_k^1\langle H_l(\boldsymbol{x}_1, t+\tau_1)\, i_n^*(\boldsymbol{x}_2, t+\tau_2)\rangle - \frac{1}{c}\frac{\partial}{\partial\tau_1}\langle E_j(\boldsymbol{x}_1, t+\tau_1)\, i_n^*(\boldsymbol{x}_2, t+\tau_2)\rangle =$$

$$= \frac{4\pi}{c}\langle i_j(\boldsymbol{x}_1, t+\tau_1)\, i_n^*(\boldsymbol{x}_2, t+\tau_2)\rangle\,,$$

*i.e.*

$$\text{(3.9)}\qquad \varepsilon_{jkl}\partial_k^1\eta_{ln}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) - \frac{1}{c}\frac{\partial}{\partial\tau_1}\gamma_{jn}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = \frac{4\pi}{c}\mathscr{J}_{jn}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2)\,.$$

This is one of the required correlation equations but it seems better not to regard it as a basic equation, because it contains the quantities $\eta_{ln}$ and $\gamma_{jn}$ which appear to be difficult to measure, although they have a well defined physical meaning. This equation is, however, useful for deriving correlation equations among $\mathscr{H}_{ln}$, $\overline{\mathscr{G}}_{ln}$, $\mathscr{G}_{jn}$, $\mathscr{E}_{jn}$ and $\mathscr{J}_{jn}$. This will be done later.

Next we multiply (3.2) by $H_m^*(\boldsymbol{x}_2, t+\tau_2)$ and set $t_1 \to t+\tau_1$ and $\partial/\partial t_1 \to \partial/\partial\tau_1$ and averaged over $t$. This leads to the relation

$$\varepsilon_{jkl}\partial_k^1\langle H_l(\boldsymbol{x}_1, t+\tau_1)\, H_m^*(\boldsymbol{x}_2, t+\tau_2)\rangle - \frac{1}{c}\frac{\partial}{\partial\tau_1}\langle E_j(\boldsymbol{x}_1, t+\tau_1)\, H_m^*(\boldsymbol{x}_2, t+\tau_2)\rangle =$$

$$= \frac{4\pi}{c}\langle i_j(\boldsymbol{x}_1, t+\tau_1)\, H_m^*(\boldsymbol{x}_2, t+\tau_2)\rangle\,,$$

*i.e.*

$$\text{(3.10}a\text{)}\qquad \varepsilon_{jkl}\partial_k^1\mathscr{H}_{lm}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) - \frac{1}{c}\frac{\partial}{\partial\tau_1}\mathscr{G}_{jm}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) =$$

$$= \frac{4\pi}{c}\eta_{mj}^*(\boldsymbol{x}_2, \boldsymbol{x}_1, \tau_2, \tau_1)\,.$$

Taking the complex conjugate of (3.2), changing $\boldsymbol{x}_1 \to \boldsymbol{x}_2$, $t_1 \to t_2$ and changing the indices $(j, k, l)$ to $(n, a, b)$ we find that

$$\varepsilon_{nab}\partial_a^2 H_b^*(\boldsymbol{x}_2, t_2) - \frac{1}{c}\frac{\partial}{\partial t_2}E_n^*(\boldsymbol{x}_2, t_2) = \frac{4\pi}{c}i_n^*(\boldsymbol{x}_2, t_2)\,.$$

We multiply the above equation by $H_l(x_1, t_1)$, set $t_1 = t+\tau_1$, $t_2 = t+\tau_2$ and average over $t$ as before. This leads to the formula

$$\varepsilon_{na}\,\partial_a^2\langle H_b^*(x_2,\, t+\tau_2)\,H_l(x_1,\, t+\tau_1)\rangle - \frac{1}{c}\frac{\partial}{\partial\tau_2}\langle E_n^*(x_2,\, t+\tau_2)\,H_l(x_1,\, t+\tau_1)\rangle =$$

$$= \frac{4\pi}{c}\langle i_n^*(x_2,\, t+\tau_2)\,H(x_1,\, t+\tau_1)\rangle\,,$$

*i.e.*

$$(3.10b)\quad \varepsilon_{nab}\,\partial_a^2\mathscr{H}_l(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c}\frac{\partial}{\partial\tau_2}\widetilde{\mathscr{G}}_{ln}(x_1, x_2, \tau_1, \tau_2) = \frac{4\pi}{c}\,\eta_{ln}(x_1, x_2, \tau_1, \tau_2)\,.$$

(3.10a) and (3.10b) give relations among $\mathscr{H}_{lm}$, $\mathscr{G}_{jm}$ and $\eta^*_{mj}$ and among $\mathscr{H}_{lb}$, $\widetilde{\mathscr{G}}_{ln}$ and $\eta_{ln}$. Therefore, if $\mathscr{H}_{lm}$ and $\mathscr{G}_{jm}$ are known from experiment the quantity $\eta^*_{mj}$ which cannot be measured directly can then be evaluated. When the field is free these two equations reduce to

$$\varepsilon_{jkl}\,\partial_k^1\mathscr{H}_m(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c}\frac{\partial}{\partial\tau_1}\mathscr{G}_{j\cdot}(x_1, x_2, \tau_1, \tau_2) = 0$$

and

$$\varepsilon_{jkl}\,\partial_k^2\mathscr{H}_{ml}(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c}\frac{\partial}{\partial\tau_2}\widetilde{\mathscr{G}}_{mj}(x_1, x_2, \tau_1, \tau_2) = 0\,,$$

respectively. These two equations are generalizations to a non-stationary field of the eqs. (I.3.12a) and (I.3.12b).

Also from (3.2) the same procedure as before leads to the following equations:

$$(3.11a)\quad \varepsilon_{jkl}\,\partial_k^1\widetilde{\mathscr{G}}_{lm}(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c}\frac{\partial}{\partial\tau_1}\mathscr{E}_{jm}(x_1, x_2, \tau_1, \tau_2) = \frac{4\pi}{c}\,\gamma^*_{mj}(x_2, x_1, \tau_2, \tau_1)\,,$$

$$(3.11b)\quad \varepsilon_{nkl}\,\partial_k^2\mathscr{G}_{jl}(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c}\frac{\partial}{\partial\tau_2}\mathscr{E}_{jn}(x_1, x_2, \tau_1, \tau_2) = \frac{4\pi}{c}\,\gamma_{jn}(x_1, x_2, \tau_1, \tau_2)$$

With the help of these equations one can evaluate the quantities $\gamma^*_{mj}$ and $\gamma_{jn}$ from the knowledge of $\mathscr{E}_{jm}$, $\widetilde{\mathscr{G}}_{lm}$ and $\mathscr{G}_{jm}$. In the case of a free field, the right-hand sides of the equations vanish and these equations are then generalizations to a non-stationary free field of eqs. (I.3.11a) and (I.3.11b).

The remaining correlation equations may be derived from eq. (3.1), which is the same as for the free field. One obtains

$$(3.12a)\quad \varepsilon_{jkl}\,\partial_k^1\mathscr{E}_{ln}(x_1, x_2, \tau_1, \tau_2) + \frac{1}{c}\frac{\partial}{\partial\tau_1}\widetilde{\mathscr{G}}_{jn}(x_1, x_2, \tau_1, \tau_2) = 0\,.$$

Making use of the symmetry properties (2.3), one readily finds from (3.12*a*) that

$$(3.12b)\qquad \varepsilon_{nab}\,\partial_a^2\mathcal{E}_{lb}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) + \frac{1}{c}\,\frac{\partial}{\partial\tau_2}\,\mathcal{G}_{ln}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 0\,.$$

One also obtains from (3.1) the equations

$$(3.13a)\qquad \varepsilon_{jkl}\,\partial_k^1\mathcal{G}_{ln}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) + \frac{1}{c}\,\frac{\partial}{\partial\tau_1}\,\mathcal{H}_{jn}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 0\,,$$

$$(3.13b)\qquad \varepsilon_{jkl}\,\partial_k^2\widetilde{\mathcal{G}}_{nl}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) + \frac{1}{c}\,\frac{\partial}{\partial\tau_2}\,\mathcal{H}_{nj}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 0\,.$$

The above four equations apply to a field whether or not it contains currents and charges. Equation (3.12*a*) gives simply the relation between $\mathcal{E}_{ln}$ and $\widetilde{\mathcal{G}}_{jn}$, and (3.12*b*) gives the relation between $\mathcal{E}_{lb}$ and $\mathcal{G}_{ln}$, and so forth. These equations are generalizations to a non-stationary field of the eqs. (I.3.9*a*), (I.3.9*b*), (I.3.10*a*) and (I.3.10*b*).

## 4. – Higher order differential equations for the correlations.

The differential equations derived so far contain derivatives of order not higher than the first. Some of the equations contain, however, the auxiliary quantities $\alpha_j$, $\beta_j$, $\gamma_{jk}$, and $\eta_{jk}$. It is possible to eliminate these quantities completely and one is then lead to differential equations of higher order, which appear to be more basic. These equations will now be derived.

Substituting (3.6*b*) into (3.5) we arrive at the second order « divergence equation » which gives the relation between $\mathcal{E}_{jk}$ and $\mathcal{P}$:

$$(4.1)\qquad \partial_j^1\partial_k^2\,\mathcal{E}_{jk}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = 16\pi^2\mathcal{P}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2)\,.$$

On multiplying (3.10*b*) by $\varepsilon_{jkl}\partial_k^1$ we obtain the equation

$$\varepsilon_{jkl}\varepsilon_{nab}\,\partial_k^1\partial_a^2\,\mathcal{H}_{lb}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) - \frac{1}{c}\,\varepsilon_{jkl}\,\partial_k^1\,\frac{\partial}{\partial\tau_2}\,\widetilde{\mathcal{G}}_{ln}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = \\ = \frac{4\pi}{c}\,\varepsilon_{jkl}\,\partial_k^1\eta_{ln}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2)\,,$$

and multiplying (3.11*b*) by $-(1/c)(\partial/\partial\tau_1)$, we find that

$$-\frac{1}{c}\,\varepsilon_{nkl}\,\partial_k^2\,\frac{\partial}{\partial\tau_1}\,\mathcal{G}_{jl}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) + \frac{1}{c^2}\,\frac{\partial}{\partial\tau_1}\,\frac{\partial}{\partial\tau_2}\,\mathcal{E}_{jn}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2) = \\ = -\frac{4\pi}{c^2}\,\frac{\partial}{\partial\tau_1}\,\gamma_{jn}(\boldsymbol{x}_1, \boldsymbol{x}_2, \tau_1, \tau_2)\,.$$

Therefore, using (3.9), it follows that

$$(4.2)\qquad \varepsilon_{jkl}\varepsilon_{nab}\partial_k^1\partial_a^2\mathscr{H}_{lb}(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c}\varepsilon_{jkl}\partial_k^1\frac{\partial}{\partial\tau_2}\tilde{\mathscr{G}}_{ln}(x_1, x_2, \tau_1, \tau_2) - \\ - \frac{1}{c}\varepsilon_{nkl}\partial_k^2\frac{\partial}{\partial\tau_1}\mathscr{G}_{jl}(x_1, x_2, \tau_1, \tau_2) + \frac{1}{c^2}\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}\mathscr{E}_{jn}(x_1, x_2, \tau_1, \tau_2) = \\ = \frac{16\pi^2}{c^2}\mathscr{I}_{jn}(x_1, x_2, \tau_1, \tau_2)\,.$$

If the field is stationary, then relations of the following form may be used:

$$\langle F(x_1, t)\,G^*(x_2, t-\tau)\rangle = \langle F(x_1, t+\tau)\,G^*(x_2, t)\rangle\,.$$

We may then replace $\partial/\partial\tau_1 \to \partial/\partial\tau$ and $\partial/\partial\tau_2 \to -\partial/\partial\tau$, since in this case we put $t_1 = t$, $t_2 = t - \tau$. Therefore eq. (4.2) becomes

$$\varepsilon_{jkl}\varepsilon_{nab}\partial_k^1\partial_a^2\mathscr{H}_{lb} + \frac{1}{c}\varepsilon_{jkl}\partial_k^1\frac{\partial}{\partial\tau}\tilde{\mathscr{G}}_{ln} - \frac{1}{c}\varepsilon_{nkl}\partial_k^2\frac{\partial}{\partial\tau}\mathscr{G}_{jl} - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\mathscr{E}_{jn} = \frac{16\pi^2}{c^2}\mathscr{I}_{jn}\,.$$

This is just the equation derived by ROMAN (7) for the stationary case.

Further we can derive a correlation equation which contains only $\mathscr{E}$ and $\mathscr{H}$. This can be done by eliminating $\mathscr{G}$ from (3.18$a$) and (3.12$b$). We apply the operator $(1/c)(\partial/\partial\tau_2)$ to eq. (3.13$a$) and obtain the relation

$$\varepsilon_{jkl}\partial_k^1\frac{1}{c}\frac{\partial}{\partial\tau_2}\mathscr{G}_{ln}(x_1, x_2, \tau_1, \tau_2) + \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathscr{H}_{jn}(x_1, x_2, \tau_1, \tau_2) = 0\,.$$

Substituting (3.12$b$) into the above equation, we find that

$$(4.3)\qquad \varepsilon_{jkl}\varepsilon_{nab}\partial_k^1\partial_a^2\mathscr{E}_{lb}(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathscr{H}_{jn}(x_1, x_2, \tau_1, \tau_2) = 0\,.$$

It should be noted that this equation contains only $\mathscr{E}$ and $\mathscr{H}$ tensors.

Also we can eliminate $\mathscr{G}$ and $\tilde{\mathscr{G}}$ from (4.2). To do this we first change the indices in (3.12$a$) as follows: $j \to l$, $k \to a$, $l \to b$. We then apply the operator $\varepsilon_{jkl}\partial_k^1$ and obtain the equation

$$\varepsilon_{jkl}\varepsilon_{lab}\partial_k^1\partial_a^2\mathscr{E}_{bn}(x_1, x_2, \tau_1, \tau_2) + \frac{1}{c}\varepsilon_{jkl}\partial_k^1\frac{\partial}{\partial\tau_1}\tilde{\mathscr{G}}_{ln}(x_1, x_2, \tau_1, \tau_2) = 0\,.$$

Utilizing the well-known identity

$$\varepsilon_{jkl}\varepsilon_{lab} = \delta_{ja}\delta_{kb} - \delta_{jb}\delta_{ka}\,,$$

where $\delta$ is the Kronecker symbol, we obtain the equation

$$(4.4)\qquad \partial_j^1\partial_k^1\mathscr{E}_{kn}(x_1, x_2, \tau_1, \tau_2) - \partial_k^1\partial_k^1\mathscr{E}_{jn}(x_1, x_2, \tau_1, \tau_2) + \frac{1}{c}\varepsilon_{jkl}\partial_k^1\frac{\partial}{\partial\tau_1}\tilde{\mathscr{G}}_{ln}(x_1, x_2, \tau_1, \tau_2) = 0 .$$

In a similar way we obtain from (3.12$b$) the relation

$$(4.5)\qquad \partial_n^2\partial_k^2\mathscr{E}_{jk}(x_1, x_2, \tau_1, \tau_2) - \partial_k^2\partial_k^2\mathscr{E}_{jn}(x_1, x_2, \tau_1, \tau_2) + \frac{1}{c}\varepsilon_{nkl}\partial_k^2\frac{\partial}{\partial\tau_2}\mathscr{G}_{jl}(x_1, x_2, \tau_1, \tau_2) = 0 .$$

Applying the operators $(1/c^2)(\partial^2/\partial\tau_2^2)$ and $(1/c^2)(\partial^2/\partial\tau_1^2)$ to the eqs. (4.6) and (4.5), respectively, we have

$$(4.4)'\qquad -\frac{1}{c^2}\varepsilon_{jkl}\partial_k^1\frac{\partial}{\partial\tau_1}\cdot\frac{\partial}{\partial\tau_2^2}\tilde{\mathscr{G}}_{ln}(x_1, x_2, \tau_1, \tau_2) = \frac{1}{c^2}\frac{\partial^2}{\partial\tau_2^2}\partial_j^1\partial_k^1\mathscr{E}_{kn}(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c^2}\frac{\partial^2}{\partial\tau_2^2}\Delta^{(1)}\mathscr{E}_{jn}(x_1, x_2, \tau_1, \tau_2) ,$$

$$(4.5)'\qquad -\frac{1}{c^2}\varepsilon_{nkl}\partial_k^2\frac{\partial}{\partial\tau_1^2}\frac{\partial}{\partial\tau_2}\mathscr{G}_{jl}(x_1, x_2, \tau_1, \tau_2) = \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1^2}\partial_n^2\partial_k^2\mathscr{E}_{jk}(x_1, x_2, \tau_1, \tau_2) - \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1^2}\Delta^{(2)}\mathscr{E}_{jn}(x_1, x_2, \tau_1, \tau_2) .$$

where $\Delta^{(i)} = \partial_k^i\partial_k^i$ $(i = 1, 2)$.

Next we apply the operator $(1/c^2)(\partial/\partial\tau_1)(\partial/\partial\tau_2)$ to (4.2). This gives

$$(4.6)\qquad \frac{1}{c^2}\varepsilon_{jkl}\varepsilon_{nab}\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}\partial_k^1\partial_a^2\mathscr{H}_{lb} - \frac{1}{c^3}\varepsilon_{jkl}\partial_k^1\frac{\partial^2}{\partial\tau_2^2}\frac{\partial}{\partial\tau_1}\tilde{\mathscr{G}}_{ln} - \frac{1}{c^3}\varepsilon_{nkl}\partial_k^2\frac{\partial^2}{\partial\tau_1^2}\frac{\partial}{\partial\tau_2}\mathscr{G}_{jl} + \frac{1}{c^4}\frac{\partial^2}{\partial\tau_1^2}\frac{\partial^2}{\partial\tau_2^2}\mathscr{E}_{jn} = \frac{16\pi^4}{c^4}\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathscr{I}_{jn} .$$

Substituting (4.4)′ and (4.5)′ into (4.6), we finally obtain the equation

$$(4.7)\qquad \frac{1}{c^2}\varepsilon_{jkl}\varepsilon_{nab}\frac{\partial}{\partial\tau_1}\frac{\partial}{\partial\tau_2}\partial_k^1\partial_a^2\mathscr{H}_{lb} + \frac{1}{c^2}\frac{\partial^2}{\partial\tau_2^2}\partial_j^1\partial_k^1\mathscr{E}_{kn} - \frac{1}{c^2}\frac{\partial^2}{\partial\tau_2^2}\Delta^{(1)}\mathscr{E}_{jn} + \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1^2}\partial_n^2\partial_k^2\mathscr{E}_{jk} - \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1^2}\Delta^{(2)}\mathscr{E}_{jn} + \frac{1}{c^4}\frac{\partial^2}{\partial\tau_1^2}\frac{\partial^2}{\partial\tau_2^2}\mathscr{E}_{jn} = \frac{16\pi^2}{c^4}\frac{\partial^2}{\partial\tau_1\partial\tau_2}\mathscr{I}_{jn} .$$

For the stationary case equation (4.7) reduces to

$$\frac{\partial^2}{\partial\tau^2}\left\{\varepsilon_{jkl}\varepsilon_{nab}\partial_k^1\partial_a^2\mathscr{H}_{lb} - \partial_j^1\partial_k^1\mathscr{E}_{kn} + \partial_k^1\partial_k^1\mathscr{E}_{jn} - \partial_n^2\partial_k^2\mathscr{E}_{jk} + \partial_k^2\partial_k^2\mathscr{E}_{jn} - \frac{1}{c^2}\frac{\partial^2}{\partial\tau^2}\mathscr{E}_{jn} - \frac{16\pi^2}{c^2}\mathscr{I}_{jn}\right\} = 0 ,$$

as can be seen by setting $\partial/\partial\tau_1 = \partial/\partial\tau$ and $\partial/\partial\tau_2 = -\partial/\partial\tau$ in (4.7). It is of interest to note that the term appearing under the differentiation sign $\partial^2/\partial\tau^2$ (*i.e.* the term in the large bracket) is identically equal to zero in the analysis of Roman relating to the stationary case.

Further we can eliminate $\mathscr{H}$ from eq. (4.7) with the help of eq. (4.3). The resultant equation is

$$
\begin{aligned}
(4.8)\qquad & \partial_k^1\partial_j^1\partial_a^2\partial_n^2\mathscr{E}_{ki} - \Delta^{(2)}\partial_j^1\partial_k^1\mathscr{E}_{kn} - \Delta^{(1)}\partial_a^2\partial_n^2\mathscr{E}_{jn} + \Delta^{(1)}\Delta^{(2)}\mathscr{E}_{jn} + \\
& + \frac{1}{c^2}\frac{\partial^2}{\partial\tau_2^2}\partial_j^1\partial_k^1\mathscr{E}_{kn} - \frac{1}{c^2}\frac{\partial^2}{\partial\tau_2^2}\Delta^{(1)}\mathscr{E}_{jn} + \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1^2}\partial_n^2\partial_k^2\mathscr{E}_{jk} - \frac{1}{c^2}\frac{\partial^2}{\partial\tau_1^2}\Delta^{(2)}\mathscr{E}_{jn} + \\
& + \frac{1}{c^4}\frac{\partial^2}{\partial\tau_1^2}\cdot\frac{\partial^2}{\partial\tau_2^2}\mathscr{E}_{jn} = \frac{16\pi^2}{c^4}\cdot\frac{\partial^2}{\partial\tau_1\,\partial\tau_2}\mathscr{J}_{jn}\,.
\end{aligned}
$$

This equation contains only the electric correlation tensor $\mathscr{E}$ and the current correlation tensor $\mathscr{J}$.

If the current correlation tensor $\mathscr{J}$ is given, the eq. (4.8) represents 9 equations for the 9 components of $\mathscr{E}$. These equations, with the subsidiary condition (4.1) make it in principle possible to determine $\mathscr{E}$ from the knowledge of $\mathscr{J}$. Once $\mathscr{E}$ is known, the magnetic correlation tenscr $\mathscr{H}$ may be determined from eqs. (4.3) subject to the «divergence relations» (3.8*a*, *b*). Finally one may determine the correlation tensors $\mathscr{G}$ and $\tilde{\mathscr{G}}$ by substituting for $\mathscr{E}$ into eqs. (3.12*a*, *b*) or by substituting for $\mathscr{H}$ into (3.13*a*, *b*).

* * *

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## RIASSUNTO (*)

Si generalizzano al caso di un campo non stazionario contenente correnti e cariche, i principali risultati derivati da E. Wolf e P. Roman, relativi alla propagazione di correlazioni di secondo ordine nei campi elettromagnetici. Si derivano le equazioni differenziali fondamentali relative alle correlazioni. Queste equazioni si dividono in due gruppi, uno dei quali contiene solo equazioni differenziali del primo ordine, ma comporta alcuni parametri difficili da determinare sperimentalmente. Se si eliminano queste quantità si ottiene un secondo gruppo di equazioni, che sono di ordine più elevato, ma contengono solo i tensori di correlazione elettrici e magnetici e un tensore che caratterizza la correlazione nelle correnti elettriche.

(*) *Traduzione a cura della Redazione.*